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Some Comments on Prime Rings

I. N. HERSTEIN*

University of Chicago, Chicago, Illinois 60637

AND

LANCE W. SMALL†

University of California, La Jolla, California 92093

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Recall that a ring R is said to be prime if the product of two of its nonzero two-sided ideals is never 0. This is equivalent to the fact that if $aRb = 0$, $a, b \in R$, then $a = 0$ or $b = 0$, that is, if $arb = 0$ for all $r \in R$ then $a = 0$ or $b = 0$.

One might very well ask if there is anything particular about the use of just two elements a and b above. For instance, is it possible that $arbrc = 0$ for all $r \in R$, R a prime ring, with elements $a \neq 0$ $b \neq 0$, $c \neq 0$ in R? More generally, is it possible to find n nonzero elements $a_1, \ldots, a_n$ in a prime ring R such that $a_1ra_2r \cdots a_{n-1}ra_n = 0$ for all $r \in R$?

One thing is certain about this last question, namely, we cannot have $a_1 = a_2 = \cdots = a_n \neq 0$. For, if this were possible, a_1R would be a nil right ideal of R of bounded index of nilpotence (namely, $n + 1$); a well-known result of Levitzki would then tell us that R has a nonzero nilpotent ideal. This is not possible in a prime ring.

Posner and Schneider addressed themselves to the questions raised in the above paragraphs in [2] and did obtain an interesting theorem about the impossibility of relations of the form $a_1ra_2r \cdots a_{n-1}ra_n = 0$ for a certain class of prime rings, and about the possibility of such relations for other prime rings. We synopsize what they did in [2]:

1. By linearizing on r and some elementary play with elements, they showed that if $arbrc = 0$ for all r in the prime ring R then one of a, b, or c must be 0.

2. For general prime rings one cannot go beyond three elements a, b, c. However, they did show that if R is a prime ring with *minimal right ideal* (and so,

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a primitive ring with minimal right ideal), and if M is a faithful, irreducible R-module, and $D = \mathrm{Hom}_R(M, M)$, then

(a) If D is infinite and $a_1 r a_2 r \cdots a_{n-1} r a_n = 0$ for all $r \in R$, then some a_i must be 0.

(b) On the other hand, if D is finite, having q elements, then one can find $q + 2$ nonzero elements $a_1, \ldots, a_{q+2}$ in R such that $a_1 r a_2 r \cdots a_{q+1} r a_{q+2} = 0$ for all $r \in R$, and that one cannot find $q + 1$ nonzero elements $a_1, \ldots, a_{q+1}$ such that $a_1 r a_2 r \cdots a_q r a_{q+1} = 0$ for all $r \in R$.

In this paper we shall push these results of Posner and Schneider much further. One could say that this whole area of results is in the nature of a curiosity and probably has no implications outside of itself. However, the results do seem to have some interest.

We begin the material with

LEMMA 1. *Let R be a prime Goldie ring and let A be the subring generated by all the regular elements of R. Then, if S is the ring of quotients of R we have:*

1. *A is a prime Goldie ring.*
2. *If $s, t \in S$ are such that $sAt = 0$ then $s = 0$ or $t = 0$.*

Proof. By Goldie's theorem, $S = D_k$, the ring of all $k \times k$ matrices over a division ring D. But then every element in S is a sum of invertible elements. Thus, given $x \in S$, $x = x_1 + \cdots + x_m$ where $x_i \in S$ are invertible. We can write $x_i = a_i b^{-1}$ where $a_i, b \in R$ and b is regular. Since each x_i is invertible in S, we have that each a_i is regular in R. Hence $x = x_1 + \cdots + x_n = (a_1 + \cdots + a_n) b^{-1}$. Since the $a_i \in A$, we see that S is an order in S. By Goldie's theorem we then know that A is a prime Goldie ring.

Suppose that $s, t \in S$ are such that $sAt = 0$. We write $s = uv^{-1}$, $t = wz^{-1}$ where u, v, w, z are in A and v, z are regular in R. If $a \in A$ then $va \in A$, hence $0 = svat = uv^{-1}vawz^{-1}$, the outcome of which is the $uAw = 0$. Because $u, v \in A$ and A is prime we obtain $u = 0$ or $w = 0$ which give, respectively, that $s = 0$ or $t = 0$.

Since the product of regular elements in R is regular, A is merely the additive group generated by all the regular elements of R.

As was pointed out to us by J. C. Robson, the ring A of Lemma 1 is actually all of R itself. The proof is not hard and is based on the Faith–Utumi theorem. However, for our purposes here, the weaker statement given in Lemma 1 suffices.

We go on to

LEMMA 2. *Let R be a prime Goldie ring and let $S = D_k$ be the ring of quotients of R. Suppose that $k \geqslant 2$.*

Let

$$e_{11} = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \end{pmatrix} \in S.$$

Then we can find $a, b \in R$, with b regular in R, such that $e_{11} = ab^{-1}$ and $aRa \neq 0$ is a domain.

Proof. Suppose the result is false; that is, whenever $e_{11} = ab^{-1}$, $a, b \in R$ then aRa is *not* a domain. Now $e_{11}Se_{11} = De_{11}$ is a division ring, and given $\delta \in D$ then $\delta e_{11} = e_{11}\delta$. Moreover, $aRa = e_{11}bRe_{11}b \subset e_{11}Se_{11}b = De_{11}b$; therefore $De_{11}b$ is *not* a domain. Hence we can find $\delta_1, \delta_2 \in D$ such that $\delta_1 e_{11}b \neq 0$ and $\delta_2 e_{11} b \neq 0$ and $(\delta_1 e_{11} b)(\delta_2 e_{11} b) = 0$. Since δ_1 is invertible and b is regular, we get that $e_{11}b\delta_2 e_{11} = 0$, and since $\delta_2 e_{11} = e_{11}\delta_2$, we end up with $e_{11}be_{11} = 0$. Because $a = e_{11}b$, we see that $ae_{11} = 0$. Thus $ae_{11} = 0$ whenever $e_{11} = ab^{-1}$. If $c \in R$ is regular then $e_{11} = acc^{-1}b^{-1} = ac(bc)^{-1}$; thus $ace_{11} = 0$. This gives us that $aAe_{11} = 0$ where A is the subring of R generated by the regular elements. Since $a \neq 0$, $e_{11} \neq 0$ are in S, by part (2) of Lemma 1 this is not possible. Therefore the lemma is proved.

We are now able to prove

THEOREM 1. *Let R be an infinite prime Goldie ring, and suppose that $a_1, \ldots, a_n \in R$ are such that $a_1ra_2r \cdots a_{n-1}ra_n = 0$ for all $r \in R$. Then some $a_i = 0$.*

Proof. R is an order in D_k where D is a division ring. If $k = 1$ then R is a domain, hence the result is trivially true. Suppose then that $k \geqslant 2$. Suppose further that each $a_i \neq 0$.

Since R is infinite, D must be infinite. Let $V = RD \subset D_k$; V is a right vector space over D. Write $e_{11} = uv^{-1}$ where $u, v \in R$ and where u has been chosen, according to Lemma 2, in such a way that uRu is a domain. We claim that there is an $x \in R$ such that $a_ixu \neq 0$ for $i = 1, 2, \ldots, n$. For, let $V_i = \{w \in V \mid a_iwu = 0\}$; we claim that V_i is a D-subvector space of V.

Since $e_{11}\delta = \delta e_{11}$ for $\delta \in D$, we have $uv^{-1}\delta = \delta uv^{-1}$, which is to say, $\delta u = uv^{-1}\delta v$. If $w \in V_i$ then $a_iwu = 0$, hence $a_iwuv^{-1}\delta v = 0$; by the above this gives $a_iw\delta u = 0$, hence $w\delta \in V_i$. Thus V_i is a right vector space over D. Moreover, $V_i \neq V$, for otherwise $a_iRu = 0$ follows, with $a_i \neq 0$, $u \neq 0$, which is impossible in a prime ring. Since the V_i are proper subspaces of V and D is infinite, V *cannot* be the set-theoretic union of $V_1, V_2, \ldots, V_n$. Hence there is a $t \in V$, $t \notin V_i$ for $i = 1, 2, \ldots, n$. Thus $a_itu \neq 0$ for $i = 1, 2, \ldots, n$. Since $t = t_1\delta_1 + \cdots + t_q\delta_q$ where $t_i \in R$, $\delta_i \in D$ we see that $a_it_ju \neq 0$ for some t_j. Let $x = t_j$; then $a_ixu \neq 0$ for $i = 1, 2, \ldots, n$ and $x \in R$. Let $b_i = a_ixu$.

Let $D_1 = v^{-1}Dv$; D_1 is a division ring lying in D_k. Let $W = D_1R$. W is a left vector space over D_1. Let $W_i = \{w \in W \mid uwb_i = 0\}$; as we did above, we get that W_i is left D_1-subspace of W. So we have that $W \neq \bigcup_{i=1}^n W_i$, since the W_i

are proper subspaces of W. As above we then get a $y \in R$ such that $uyb_i \neq 0$ for $i = 1, 2, \ldots, n$. Writing this out we have that $uya_ixu \neq 0$ for $i = 1, 2, \ldots, n$ with $x, y \in R$.

Since $a_1ra_2r \cdots a_{n-1}ra_n = 0$ for all $r \in R$, replace r in this by $xuruy$, premultiply the resulting relation by uy and postmultiply by xu. We get

$$(uya_1xu)\, r(uya_2xu) \cdots (uya_{n-1}xu)\, r(uya_nxu) = 0.$$

Since the $uya_ixu \neq 0$ are in uRu, a *domain*, picking $r \neq 0 \in uRu$, we get a contradiction. With this the theorem is proved.

The theorem has two immediate consequences, namely, for rings with descending chain condition on right ideals—that is, Artinian rings—and for those with ascending chain conditions on right ideals—that is, right Neotherian rings. In each case such a prime ring is a prime Goldic ring. Hence,

COROLLARY 1. *Let R be an infinite prime Artinian ring (hence, an infinite simple Artinian ring). If $a_1, \ldots, a_n \in R$ are such that $a_1ra_2r \cdots a_{n-1}ra_n = 0$ for all $r \in R$, then some $a_i = 0$.*

COROLLARY 2. *Let R be an infinite (right) Noetherian prime ring. If $a_1, \ldots, a_n \in R$ are such that $a_1ra_2r \cdots a_{n-1}ra_n = 0$ for all $r \in R$, then some $a_i = 0$.*

Of course, Corollary 1 is a special case of the result of Posner and Schneider.

We can pass from the case of simple Artinian rings in Corollary 1 to general simple rings. However, we must pay a small price, namely, we must insist that the ring have a unit element. This is

THEOREM 2. *Let R be an infinite simple ring with unit element and suppose that $a_1, \ldots, a_n$ in R are such that $a_1ra_2 \cdots a_{n-1}ra_n = 0$ for all $r \in R$. Then some $a_i = 0$.*

Proof. Suppose that none of the $a_i = 0$. Then R satisfies the nontrivial generalized polynomial identity $a_1xa_2x \cdots a_{n-1}xa_n$. By a result of Martindale, RC is a primitive ring with minimal right ideal, where C is the extended centroid of R (see [1, pp. 20–31 and Theorem 1.3.2]). However, since R is a simple ring with unit element, C is merely the center of R, and RC is R itself. So R is a simple ring with minimal right ideal and with a unit element. But then R is Artinian. By Corollary 1 to Theorem 1, we have that some $a_i = 0$, in contradiction to $a_i \neq 0$ for all i. With this the theorem is proved.

One cannot drop the assumption of a unit element in Theorem 2, as we already know from the work of Posner and Schneider. To be explicit, if R is the ring of all infinite matrices (α_{ij}) where $\alpha_{ij} \in F$, the field of two elements. such that all but a finite number of the α_{ij} are zero, then R is simple. However, the elements $a_1 = e_{11}$, $a_2 = e_{11}$, $a_3 = e_{11} + e_{21}$, and $a_4 = e_{22}$ satisfy $a_1ra_2ra_3ra_4 = 0$ for all $r \in R$.

We close the paper with another theorem which assures us of the impossibility of $a_1ra_2r \cdots a_{n-1}ra_n = 0$ for all $r \in R$ with nonzero a_i, in a prime ring R. This is

THEOREM 3. *Let R be a prime ring such that its extended centroid C is infinite. If $a_1, \dots, a_n \in R$ are such that $a_1ra_2r \cdots a_{n-1}ra_n = 0$ for all $r \in R$, then some $a_i = 0$.*

Proof. Suppose that all the $a_i \neq 0$. Then R satisfies the nontrivial generalized polynomial identity. Hence, if C is the extended centroid of R, $S = RC$ is a primitive ring with minimal right ideal eS, $e^2 = e$, where $eSe = De$, D a division ring, by the previously cited theorem of Martindale. Moreover, C is the center of D (and D is finite-dimensional over C).

By the construction of S there exists a nonzero ideal U of R such that $eU \neq 0 \subset R$. Since eU is a right ideal of R and cannot be nilpotent, $eUe \neq 0$. So there is a $u \in U$ such that $\delta e = eue \neq 0$, where $\delta \in D$. Moreover, since U is an ideal of R and $eu \in R$, by the primeness of R, $euU \neq 0$. Because euU is a nonzero right ideal of R, it, too, cannot be nilpotent. Thus $euUe \neq 0$, hence there is a $v \in U$ such that $euve = \alpha e \neq 0$ where $\alpha \in D$.

We argue in a vein similar to that of the proof of Theorem 1. Let $S_i = \{t \in S \mid a_ite = 0\}$; since C is in the center of S, S_i is a vector space over$C$. By the primeness of S, $S_i \neq S$ for each i, and since C is an infinite field, $S \neq \bigcup S_i$. This way we get an element of S, and from it an element of R, such that $b_i = a_ixe \neq 0$ for all $i = 1, 2, \dots, n$, with $x \in R$. Letting $T_i = \{y \in R \mid euyb_i = 0\}$, T_i is a subspace of S over C, so we get a $y \in R$ such that $euyb_i \neq 0$ for $i = 1, 2, \dots, n$. Hence if $c_i = euya_ixe$ then $c_i \neq 0$ for $i = 1, 2, \dots, n$; because $c_i \in eSe$ a division ring, and since $0 \neq eue = \delta e$, we get $c_ieur = c_iue \neq 0$, whence $c_iu \neq 0$. Moreover, because eu, x, y, a_i are all in R, $d_i = euya_ixeu \in R$. In

$$a_1ra_2r \cdots a_{n-1}ra_n = 0$$

replace r by $xeureuy$. In this way we get that $d_1rd_2r \cdots d_{n-1}rd_n = 0$ for all $r \in R$. Put $r = v$ where v is the element we produced earlier in R such that $euve = \alpha e \neq 0$, $\alpha \in D$. Now $d_i = euya_ixeu = \delta_ieu \neq 0$, where $\delta_i \in D$. So, since $e\delta_ie = \delta_ie$,

$$\begin{aligned} 0 = d_1v\, d_2v \cdots d_{n-1}v\, d_n &= \delta_1euv\, \delta_2euv \cdots \delta_{n-1}euv\, \delta_n \\ &= \delta_1euve\, \delta_2euve\, \delta_3 \cdots euve\, \delta_n = \delta_1\alpha\, \delta_2\alpha \cdots \alpha\, \delta_n . \end{aligned}$$

All the elements $\alpha, \delta_1, \dots, \delta_n$ are nonzero and in the division ring D, yet their product is 0. With this contradiction we have proved the theorem.

It is not clear what hypothesis on a prime ring R will guarantee that its extended centroid is infinite. One such condition is that the centroid of R be infinite. However, it is easy enough to give examples of prime rings whose centroid is finite but whose extended centroid is infinite.

COROLLARY. *Let R be a prime ring whose centroid is infinite. If $a_1, \ldots, a_n \in R$ are such that $a_1 r a_2 r \cdots a_{n-1} r a_n = 0$ for all $r \in R$ then some $a_i = 0$.*

If R is of characteristic 0 then its centroid is certainly infinite. So, in such a ring we cannot have $a_1 r a_2 r \cdots a_{n-1} r a_n = 0$ for all $r \in R$, with the $a_i \neq 0$.

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